CONCIRCULARLY Φ-RECURRENT LORENTZIAN Α-SASAKIAN MANIFOLDS ENDOWED WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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ABSTRACT

This paper focuses on the investigation of concircularly φ-recurrent Lorentzian α-Sasakian manifolds equipped with a semi-symmetric non-metric connection.

Keywords and Phrases: *Concircularly φ-symmetric manifold; concircularly φ-recurrent manifold; n-Einstein manifold.*

INTRODUCTION

The concept of local symmetry in Riemannian geometry has been progressively weakened and generalized by several researchers in order to study broader classes of manifolds with richer geometric structures. One such weaker form of local symmetry was introduced by Takahashi, who defined the notion of local φ-symmetry in the context of Sasakian manifolds. This development motivated further generalizations in contact metric geometry. Extending the idea of φ-symmetry, De introduced the concept of φ-recurrent Sasakian manifolds, which provides a natural framework for examining curvature recurrence conditions governed by the structure tensor φ. These notions have played a significant role in understanding the geometric behavior of manifolds endowed with additional structure, particularly in Lorentzian settings. Parallel to these developments, Fridmann and Schouten introduced the notion of a semi- symmetric linear connection on differentiable manifolds, thereby relaxing the classical requirement of symmetry of the affine connection. Subsequently, Hayden studied metric connections with torsion on Riemannian manifolds, highlighting their geometric and physical relevance. Further investigations by Yano and Golab systematically analyzed semi-symmetric and quarter-symmetric connections in the framework of affine geometry. Later, several authors, including De, Sharfuddin and Hussain, Rastogi, Mishra and Pandey, and Bagewadi, explored various curvature properties and geometric implications of semi-symmetric connections on different types of manifolds.

Motivated by these studies, the present work is devoted to the investigation of concircularly φ-recurrent Lorentzian α-Sasakian manifolds equipped with a semi-symmetric non-metric connection. The main objective is to analyze the curvature behavior induced by such a connection and to examine its influence on the underlying geometric structure. It is shown that a φ-recurrent Lorentzian α-Sasakian manifold admitting a semi-symmetric non-metric connection necessarily satisfies the condition of being an *n*-Einstein manifold. Furthermore, it is established that the characteristic vector field ξ and the vector field associated with the 1- form *A*are co-directional. Finally, we prove that a concircularly φ-recurrent Lorentzian α- Sasakian manifold with a semi-symmetric non-metric connection is an *n*-Einstein manifold.

Preliminaries

Let M be an n-dimensional differentiable manifold. The manifold M is said to be a Lorentzian α-Sasakian manifold if it admits a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a 1-form η, and a Lorentzian metric gsatisfying the following relations for all vector fields X, Y ∈ TM [2, 3, 13]:

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1,$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), g(X, \xi) = \eta(X),$$

$$\phi\xi = 0, \eta(\phi X) = 0,$$

$$(D_X\phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\},$$

where Ddenotes the covariant differentiation with respect to the Lorentzian metric g. In addition, a Lorentzian α-Sasakian manifold satisfies the following identities:

$$(D_X\xi)Y = \alpha\phi X,$$

$$(D_X\eta)Y = -\alpha g(\phi X, Y).$$

Furthermore, the curvature tensor R of a Lorentzian α-Sasakian manifold obeys the following relations [2, 3, 12]

$$R(X, Y)\xi = \alpha^2\{\eta(Y)X - \eta(X)Y\},$$

$$R(\xi, X)Y = \alpha^2\{g(X, Y)\xi - \eta(Y)X\},$$

$$R(\xi, X)\xi = \alpha^2\{\eta(X)\xi + X\},$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X),$$
$$\eta(R(X,Y)Z) = \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\},$$
$$g(R(\xi,X)Y,\xi) = -\alpha^2\{g(X,Y) + \eta(X)\eta(Y)\}.$$

Here, S denotes the Ricci tensor and Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y).$$

A Lorentzian α-Sasakian manifold Mis called an η-Einstein manifold if its Ricci tensor satisfies

$$S(X,Y) = a\, g(X,Y) + b\, \eta(X)\eta(Y),$$

for all vector fields X, Y, where aand bare smooth functions on M. In the special case when $b = 0$, the manifold reduces to an Einstein manifold.

It has been shown in [2, 8] that if a Lorentzian α-Sasakian manifold is an η-Einstein manifold, then the functions aand bsatisfy the relation

$$a + b = -\alpha^2(n-1).$$

Definition 1: A Lorentzian α-Sasakian manifold Mis said to be locally **ϕ**-symmetric if the curvature tensor Rsatisfies

$$\phi^2((D_W R)(X,Y)Z) = 0,$$

for all vector fields $X, Y, Z, W \in TM$.

Definition 2: A Lorentzian α-Sasakian manifold Mis called **ϕ**-recurrent if there exists a non-zero 1-form A such that

$$\phi^2((D_W R)(X,Y)Z) = A(W)\, R(X,Y)Z,$$

for all $X, Y, Z, W \in TM$. Here, the 1-form A is defined by

$A(W) = g(W,p)$, where p is the vector field metrically associated with A.

Definition 3: A Lorentzian α-Sasakian manifold Mis said to be concircularly **ϕ**-recurrent if there exists a non-zero 1-form A such that

$$\phi^2((D_W C)(X,Y)Z) = A(W)\, C(X,Y)Z,$$

where Cdenotes the concircular curvature tensor defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}.$$

Here, Ris the Riemannian curvature tensor and rdenotes the scalar curvature of the manifold. A linear connection Don an n-dimensional differentiable manifold is called a semi-symmetric connection if its torsion tensor Tsatisfies [1, 4, 7]

$$T(X,Y) = D_X Y - D_Y X - [X,Y] = \eta(Y)X - \eta(X)Y,$$

for all vector fields $X, Y \in TM$.

A semi-symmetric connection Dis said to be a semi-symmetric non-metric connection if it does not preserve the metric, that is, $D_X g \neq 0$, for some vector field X on M [8].

Lorentzian α-Sasakian Manifold with a Semi-Symmetric Non-Metric Connection

Let M be a Lorentzian α-Sasakian manifold endowed with a semi-symmetric non-metric connection $\bar{D}$. Such a connection can be defined in terms of the Levi-Civita connection D by the relation

$$\bar{D}_X Y = D_X Y + \eta(Y)X.$$

For this connection, the covariant derivative of the metric tensor satisfies

$$(\bar{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(Y,X).$$

The connection $\bar{D}$ defined by above equations is therefore referred to as a semi-symmetric non-metric connection on a Lorentzian α-Sasakian manifold.

Let $\bar{R}$ and R denote the curvature tensors corresponding to the connections $\bar{D}$ and D, respectively. The curvature tensors are related by

$$\bar{R}(X,Y)Z = R(X,Y)Z - \alpha g(\phi X, Z)Y - \alpha g(\phi Y, Z)X.$$

From equation, the Ricci tensors $\bar{S}$ and S associated with the connections $\bar{D}$ and D, respectively, are related by

$$\bar{S}(Y,Z) = S(Y,Z) + \alpha(n-1)g(\phi Y, Z).$$

Taking the trace of equation, we obtain $\bar{r} = r$,

where $\bar{r}$ and r denote the scalar curvatures corresponding to the connections $\bar{D}$ and D, respectively.

Recurrent Lorentzian α-Sasakian Manifold with a Semi-Symmetric Non-Metric Connection: Analogous to Definition 2, a Lorentzian α-Sasakian manifold M is said to be **ϕ**-recurrent with respect to a semi-symmetric non-metric connection if the curvature tensor $\bar{R}$ of the connection $\bar{D}$ satisfies

$$\phi^2((\bar{D}_W\bar{R})(X,Y)Z) = A(W)\bar{R}(X,Y)Z, \qquad \text{where A is a non-zero 1-form.}$$

Using relation in equation, we obtain

$$(\bar{D}_W\bar{R})(X,Y)Z + \eta((\bar{D}_W\bar{R})(X,Y)Z)\xi = A(W)\bar{R}(X,Y)Z.$$

Taking the inner product of equation with an arbitrary vector field U, we have

$$g((\bar{D}_W\bar{R})(X,Y)Z,U) + \eta((\bar{D}_W\bar{R})(X,Y)Z)g(\xi,U) = A(W)g(\bar{R}(X,Y)Z,U).$$

Let $\{e_i\}$, $i = 1,2,\dots,n$, be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in equation and summing over i, $1 \le i \le n$, we obtain

$$(\bar{D}_W\bar{S})(Y,Z) + \eta((\bar{D}_W\bar{R})(e_i,Y)Z)\eta(e_i) = A(W)\bar{S}(Y,Z).$$

Now, putting $Z = \xi$ in equation, the second term reduces to

$$g((\bar{D}_W\bar{R})(e_i,Y)\xi,\xi) = 0.$$

Hence, equation yields

$$(\bar{D}_W\bar{S})(Y,\xi) = A(W)\bar{S}(Y,\xi).$$

Using the definition of the covariant derivative of the Ricci tensor, we have

$$(\bar{D}_W\bar{S})(Y,\xi) = W(\bar{S}(Y,\xi)) - \bar{S}(\bar{D}_WY,\xi) - \bar{S}(Y,\bar{D}_W\xi).$$

Applying relations above, and in equation, we obtain

$$\begin{aligned}(\bar{D}_W\bar{S})(Y,\xi) = {} & \alpha S(Y,\phi W) + S(Y,W) - \alpha(\alpha+1)(n-1)g(Y,\phi W) \\ & -\alpha^2(n-1)g(Y,W) + \alpha^2(n-1)g(\phi Y,\phi W).\end{aligned}$$

In view of equations above, we get

$$\begin{aligned}\alpha S(Y,\phi W) + S(Y,W) & -\alpha(\alpha+1)(n-1)g(Y,\phi W) \\ & -\alpha^2(n-1)g(Y,W) + \alpha^2(n-1)g(\phi Y,\phi W) = \alpha^2(n-1)A(W)\eta(Y).\end{aligned}$$

Replacing Y by **ϕ** Y in the above equation, we obtain

$$\begin{aligned}\alpha S(\phi Y,\phi W) + S(\phi Y,W) & -\alpha(\alpha+1)(n-1)g(\phi Y,\phi W) \\ & -\alpha^2(n-1)g(\phi Y,W) + \alpha^2(n-1)g(Y,\phi W) = 0.\end{aligned}$$

Interchanging Y and W in equation, we get

$$\begin{aligned}\alpha S(\phi W,\phi Y) + S(\phi W,Y) & -\alpha(\alpha+1)(n-1)g(\phi W,\phi Y) \\ & -\alpha^2(n-1)g(W,Y) + \alpha^2(n-1)g(W,\phi Y) = 0.\end{aligned}$$

Adding equations and simplifying, we obtain

$$S(Y, \phi W) = (\alpha^2 + 1)(n - 1)g(\phi Y, \phi W).$$

Using relations above, and these further yields

$$S(Y, W) = (\alpha^2 + 1)(n - 1)g(Y, W) + (n - 1)\eta(Y)\eta(W).$$

This results are completely proof.

CONCLUSION

In this work, we have investigated the geometric properties of ϕ-recurrent Lorentzian α-Sasakian manifolds endowed with a semi-symmetric non-metric connection. By extending the notion of ϕ-recurrence to the setting of semi-symmetric non-metric connections, we established a clear relationship between the curvature structure induced by such connections and the intrinsic geometry of Lorentzian α-Sasakian manifolds. A principal outcome of the study is the proof that a ϕ-recurrent Lorentzian α-Sasakian manifold with a semi-symmetric non-metric connection necessarily satisfies the η-Einstein condition. This result highlights the strong restriction imposed by the ϕ-recurrence condition on the Ricci tensor and demonstrates that the curvature behavior of the manifold is tightly controlled by the contact structure and the chosen connection. Moreover, it has been shown that the characteristic vector field ξand the vector field associated with the 1-form Aare co-directional, which further emphasizes the alignment between the recurrence form and the underlying geometric structure.

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